Consider the region bounded by $x = y^2$ and $y = x - 2$. **SCORE: _____ / 45 PTS**

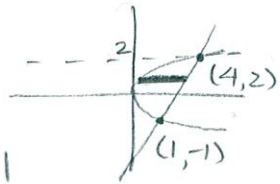
$y^2 = y + 2$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$y = 2, -1$

$x = 4, 1$



[a] Find the volume if the region is revolved around the line $y = 2$.
Your final answer must be a number, not an integral.

$$4 \quad \boxed{\int_{-1}^{2} \underbrace{2\pi}_{3}\, \overbrace{(2-y)}^{4}\overbrace{(y+2-y^2)}^{4}\, dy}$$

$$= 2\pi \int_{-1}^{2} \underbrace{(y^3 - 3y^2 + 4)}\, dy \quad 4$$

$$= 2\pi \underbrace{\left(\tfrac{1}{4}y^4 - y^3 + 4y\right)}\Big|_{-1}^{2} \quad 4$$

$$= 2\pi \left(\tfrac{1}{4}(16-1) - (8 - -1) + 4(2 - -1)\right)$$

$$= 2\pi \left(\tfrac{15}{4} - 9 + 12\right)$$

$$= 2\pi \cdot \underbrace{\tfrac{27}{4}}_{4} = \underbrace{\tfrac{27\pi}{2}} \quad 3$$

[b] If you used the disk or washer method in [a],
write, **BUT DO NOT EVALUATE**, an integral (or sum of integrals) for the same volume using the shell method.
If you used the shell method in [a],
write, **BUT DO NOT EVALUATE**, an integral (or sum of integrals) for the same volume using the disk or washer method.

$$2 \quad \int_{0}^{1} \pi \left(\overbrace{(2 - -\sqrt{x})^2}^{2\frac{1}{2}} - \overbrace{(2-\sqrt{x})^2}^{2} \right) dx$$

$$2 \quad + \int_{1}^{4} \underbrace{\pi}_{2} \left(\underbrace{(2-(x-2))^2}_{2\frac{1}{2}} - \underbrace{(2-\sqrt{x})^2}_{2} \right) dx$$

A solid of revolution has volume $\int_1^4 \pi((10+\sqrt{y})^2-(10-2y)^2)\,dy$. Find the equation of the axis of revolution, **SCORE: _____ / 15 PTS**

and the equations of the boundaries of the region being revolved. Sketch the region being revolved.

<u>Do NOT use the *x*– nor *y*–axes as boundaries nor the axis of revolution.</u>

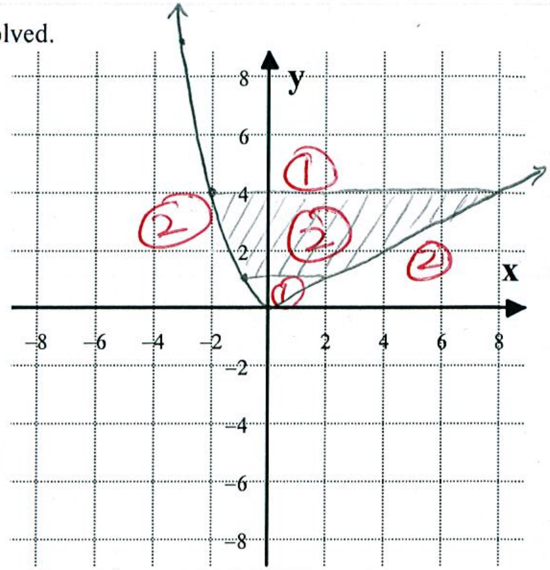
Equation of axis of revolution: 2 $x=10$

Equations of boundaries: 1 $y=1$

1 $y=4$

2 $x=-\sqrt{y}$ OR $y=x^2\ (x\le 0)$

1 $x=2y$ OR $y=\frac{1}{2}x$



Find the ***y*-coordinate** of the centroid of the region bounded by $y = e^x$, $y = e^{-x}$ and $x = 1$. **SCORE: _____ / 20 PTS**

Your final answer must be a number, not an integral.

$$A = \int_0^1 (e^x - e^{-x})\,dx$$ 3

$$= (e^x + e^{-x})\Big|_0^1$$ 2

$$= e + \frac{1}{e} - (1+1)$$ 2

$$= \frac{e^2 + 1 - 2e}{e}$$

$$= \frac{(e-1)^2}{e}$$

$$\bar{y} = \frac{e}{2(e-1)^2} \int_0^1 (e^{2x} - e^{-2x})\,dx$$ 3 3

$$= \frac{e}{2(e-1)^2} \left(\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}\right)\Big|_0^1$$ 2

$$= \frac{e}{2(e-1)^2} \cdot \frac{1}{2}\left(e^2 + e^{-2} - 2\right)$$ 2

$$= \frac{\cancel{e}}{4(e-1)^2} \cdot \frac{e^4 + 1 - 2e^2}{e^{\cancel{2}}}$$

$$= \frac{(e^2-1)^2}{4e(e-1)^2}$$

$$= \frac{(e+1)^2}{4e}$$ 3

Find the length of the parametric curve $\begin{array}{l} x = e^{-2t}\cos 3t \\ y = e^{-2t}\sin 3t \end{array}$ over $0 \le t \le \dfrac{\pi}{2}$.

SCORE: _____ / 25 PTS

<u>**Your final answer must be a number, not an integral.**</u>

$$\frac{dx}{dt} = -2e^{-2t}\cos 3t - 3e^{-2t}\sin 3t$$

$$\frac{dy}{dt} = -2e^{-2t}\sin 3t + 3e^{-2t}\cos 3t$$

4

$$\int_0^{\frac{\pi}{2}} \left[\underbrace{(-2e^{-2t}\cos 3t - 3e^{-2t}\sin 3t)^2}_{4} + \underbrace{(3e^{-2t}\cos 3t - 2e^{-2t}\sin 3t)^2}_{4}\right]^{\frac{1}{2}} dt$$

$$= \int_0^{\frac{\pi}{2}} (4e^{-4t}\cos^2 3t + 12e^{-4t}\cos 3t\sin 3t + 9e^{-4t}\sin^2 3t + 4e^{-4t}\sin^2 3t - 12e^{-4t}\cos 3t\sin 3t + 9e^{-4t}\cos^2 3t)^{\frac{1}{2}} dt$$

4

$$= \int_0^{\frac{\pi}{2}} (4e^{-4t} + 9e^{-4t})^{\frac{1}{2}} dt$$

$$= \int_0^{\frac{\pi}{2}} \underbrace{\sqrt{13}\,e^{-2t}}_{3} dt$$

$$= \underbrace{-\frac{\sqrt{13}}{2}e^{-2t}}_{3}\Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{\sqrt{13}}{2}(e^{-\pi} - 1) = \underbrace{\frac{\sqrt{13}}{2}(1 - e^{-\pi})}_{3}$$

JJ rents out bedrooms in her house to college students. Each student is required to leave a security deposit when they start renting from her. When they stop renting, some amount of the security deposit is given back to the student, depending on the condition of the room when they leave. Let X be the proportion (fraction) of the security deposit returned to a randomly selected renter. Find the mean proportion of security deposit returned if the probability density function of X has the form **SCORE: _____ / 20 PTS**

$$f(x) = \begin{cases} \dfrac{k}{1+x^2}, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

Your final answer must be a number, not an integral.

$$\underbrace{\int_0^1 \frac{k}{1+x^2}\,dx = 1}_{4}$$

$$k\underbrace{\tan^{-1}x}_{2}\Big|_0^1 = 1$$

$$\underbrace{k\left(\frac{\pi}{4}\right)}_{2} = 1$$

$$\underbrace{k = \frac{4}{\pi}}_{1}$$

$$4 \quad \underbrace{\frac{4}{\pi}\int_0^1 \frac{x}{1+x^2}\,dx}$$

$$u = 1 + x^2 \quad x = 1 \rightarrow u = 2 \quad x = 0 \rightarrow u = 1$$

$$du = 2x\,dx$$

$$\tfrac{1}{2}du = dx$$

$$= \frac{4}{\pi}\int_1^2 \frac{1}{2}\frac{1}{u}\,du$$

$$= \underbrace{\frac{2}{\pi}}_{2}\,\underbrace{\ln|u|}_{2}\Big|_1^2$$

$$= \frac{2}{\pi}(\ln|2| - \ln|1|)$$

$$= \underbrace{\frac{2}{\pi}\ln 2}_{3}$$

Find the area of the region between $y = \sin x$ and $y = \sin 2x$ on $[0, \pi]$. **SCORE: _____ / 25 PTS**

Your final answer must be a number, not an integral.

$$\sin x = 2\sin x \cos x$$

$$0 = \sin x(2\cos x - 1)$$

3 EACH EXCEPT AS NOTED

$$\sin x = 0 \rightarrow x = 0, \pi$$

$$\cos x = \tfrac{1}{2} \rightarrow x = \tfrac{\pi}{3}$$

$$3\tfrac{1}{2} \quad \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)\,dx + \int_{\frac{\pi}{3}}^{\pi} (\sin x - \sin 2x)\,dx \quad 3\tfrac{1}{2}$$

$$= \left(-\tfrac{1}{2}\cos 2x + \cos x\right)\Big|_0^{\frac{\pi}{3}} + \left(-\cos x + \tfrac{1}{2}\cos 2x\right)\Big|_{\frac{\pi}{3}}^{\pi}$$

$$= \left(-\tfrac{1}{2}\left(-\tfrac{1}{2}\right) + \tfrac{1}{2} - \left(-\tfrac{1}{2}(1) + 1\right) + \left(-(-1) + \tfrac{1}{2}(1) - \left(-\tfrac{1}{2} + \tfrac{1}{2}\left(-\tfrac{1}{2}\right)\right)\right)\right)$$

$$= \left(\tfrac{1}{4} + \tfrac{1}{2} - \left(-\tfrac{1}{2} + 1\right)\right) + \left(1 + \tfrac{1}{2} - \left(-\tfrac{1}{2} - \tfrac{1}{4}\right)\right)$$

$$= \tfrac{1}{4} + \tfrac{9}{4} = \tfrac{5}{2}$$